Decompositions of Abelian surface and quadratic forms

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Abstract

When a complex Abelian surface can be decomposed into a product of two elliptic curves, how many decompositions does the Abelian surface admit ? We provide arithmetic formulae for the number of decompositions of a complex Abelian surface.

1 Introduction and main results

Throughout this paper, an *Abelian surface* means a complex Abelian surface.

Let A be an Abelian surface which can be decomposed into a product of two elliptic curves. In general, the choice of a decomposition of A is not unique even up to isomorphism. In the present paper we study the number of decompositions of A. For this problem, there are pioneering works of Hayashida [5] and Shioda-Mitani [13] : Let $\rho(A)$ be the Picard number of A and let T_A be the transcendental lattice of A, which is the orthogonal complement of the Néron-Severi lattice in $H^2(A,\mathbb{Z})$. When $\rho(A)=4$ and T_A is primitive, Shioda and Mitani, with the cooperation of Hirzebruch, expressed the number of decompositions of A in terms of the class number of a certain imaginary quadratic order determined by A. On the other hand, Hayashida calculated the number of decompositions when $\rho(A)=3$, in connection with the number of principal polarizations.

It is natural to expect counting formulae for the decompositions for all decomposable Abelian surface, which complete the works of Hayashida and Shioda-Mitani. The purpose of this paper is to give such counting formulae uniformly by a lattice-theoretic method. Firstly we give precise definitions.

Definition 1.1. Let A be an Abelian surface.

(1) A *decomposition* of A is an ordered pair (E_1, E_2) of elliptic curves in A such that the natural homomorphism $E_1 \times E_2 \to A$ is an isomorphism. The Abelian surface A is *decomposable* if there exists a decomposition of A.

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(2) Two decompositions (E_1, E_2) and (F_1, F_2) of A are *strictly isomorphic* if $E_1 \simeq F_1$ and $E_2 \simeq F_2$, or equivalently if there exists an automorphism f of A such that $f(E_i) = F_i$. Two decompositions (E_1, E_2) and (F_1, F_2) of A are *isomorphic* if (E_1, E_2) is strictly isomorphic to (F_1, F_2) or to (F_2, F_1).

There are known several criterions for an Abelian surface to be decomposable. For example, Abelian surfaces with Picard number 4 are always decomposable [13]. Let $\mathrm{Dec}(A)$ (resp. $\widetilde{\mathrm{Dec}}(A)$) be the set of isomorphism (resp. strictly isomorphism) classes of decompositions of A, and put

$$\delta(A) := |\,\mathrm{Dec}(A)\,| \quad \text{and} \quad \widetilde{\delta}(A) := |\,\widetilde{\mathrm{Dec}}(A)\,|. \tag{1.1}$$

The number $\delta(A)$ is regarded as the number of decompositions of A, while $\widetilde{\delta}(A)$ is considered as the number of decompositions counted with multiplicity. If we define

$$\delta_0(A) := \Big|\,\{E : \text{elliptic curve},\ E \times E \simeq A\}/\simeq\,\Big|, \tag{1.2}$$

an obvious relation

$$\widetilde{\delta}(A) = 2\delta(A) - \delta_0(A) \tag{1.3}$$

holds. Hence the knowledge of $\widetilde{\delta}(A)$ in addition to that of $\delta(A)$ would enable us to study the decompositions of A more closely.

We shall express the numbers $\delta(A)$ and $\widetilde{\delta}(A)$ in terms of the arithmetic of the transcendental lattice T_A. Let $\mathcal{G}(T_A)$ be the genus of T_A, i.e., the set of isometry classes of lattices isogenus to T_A. For an even lattice T, let D_T be the discriminant form of T, which is a finite quadratic form associated to T. We have a natural homomorphism $O(T) \to O(D_T)$ between the isometry groups. For a natural number $n > 1$, let $\tau(n)$ be the number of the prime divisors of n. We put $\tau(1) := 1$. Our formula for $\delta(A)$ is stated as follows.

Theorem 1.2. *Let A be a decomposable Abelian surface. Then $2 \leq \rho(A) \leq 4$ and the decomposition number $\delta(A)$ is given as follows.*

(1) *When $\rho(A) = 2$, one has $\delta(A) = 1$.*

(2) *When $\rho(A) = 3$, one has $\delta(A) = 2^{\tau(N)-1}$ where $2N = -\det(T_A)$.*

(3) *When $\rho(A) = 4$ and $T_A \not\simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}, \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}, n > 1$, one has*

$$\delta(A) = \sum_{T \in \mathcal{G}(T_A)} |\,O(D_T)/O(T)\,|.$$

(4) *When $\rho(A) = 4$ and $T_A \simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}$ or $\begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}, n > 1$, one has*

$$\delta(A) = \begin{cases} (2^{-4} + 2^{-\tau(n)-3}) \cdot |O(D_{T_A})| & \text{if } T_A \simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}, \\ 3^{-2} \cdot (2^{-2} + 2^{-\tau(n)}) \cdot |O(D_{T_A})| & \text{if } T_A \simeq \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}, n : \text{odd}, \\ 3^{-2} \cdot (2^{-2} + 2^{-\tau(2^{-1}n)}) \cdot |O(D_{T_A})| & \text{if } T_A \simeq \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}, n : \text{even}. \end{cases}$$

On the other hand, the number $\widetilde{\delta}(A)$ is expressed in slightly different way (in the case of $\rho(A) = 4$). The set of proper equivalence classes of *oriented* lattices isogenus to T_A is denoted by $\widetilde{\mathcal{G}}(T_A)$.

Theorem 1.3. *Let A be a decomposable Abelian surface. The strict decomposition number $\widetilde{\delta}(A)$ is given as follows.*

(1) *If $\rho(A) = 2$, then $\widetilde{\delta}(A) = 2$.*

(2) *If $\rho(A) = 3$, then*

$$\widetilde{\delta}(A) = \begin{cases} 1, & N = 1, \\ 2^{\tau(N)}, & N > 1, \end{cases}$$

where $2N = -\det(T_A)$.

(3) *If $\rho(A) = 4$ and $T_A \not\simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}, \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}, n > 1$, then*

$$\widetilde{\delta}(A) = 2^{-1} \cdot |\widetilde{\mathcal{G}}(T_A)| \cdot |O(D_{T_A})|.$$

(4) *If $\rho(A) = 4$ and $T_A \simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}$ or $\begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}, n > 1$, then $\widetilde{\delta}(A) = 2\delta(A)$.*

The number $|O(D_{T_A})|$ appearing in the case of $\rho(A) = 4$ is calculated explicitly in Section 6. On the other hand, the numbers $|\widetilde{\mathcal{G}}(T_A)|$ and $|\mathcal{G}(T_A)|$ are rather deep and classical quantities. The reader may consult to [4] for the calculations of these quantities.

When $\rho(A) = 4$ and T_A is primitive, we have two types of formulae for $\delta(A)$ (or for $\widetilde{\delta}(A)$) : Shioda and Mitani's ideal-theoretic formula ([13], see Theorem 5.10) and our lattice-theoretic formula. These two formulae are unified by the classical relation between primitive binary forms and quadratic fields. In particular, the comparison of two formulae will lead to an expression of the number of genera in a class group in terms of the discriminant form (Corollary 5.11).

The counting formula in the case of $\rho(A) = 3$ is known to Hayashida [5], who calculated $\delta(A)$ as the number of reducible principal polarizations. The number N is defined in a different way in [5]. Given a decomposition (E_1, E_2) of A with $\rho(A) = 3$, we will determine explicitly all other members of $\widetilde{\mathrm{Dec}}(A)$ from (E_1, E_2). It turns out that the periods of the members of $\widetilde{\mathrm{Dec}}(A)$, defined as points of the modular curve $\Gamma_0(N)\backslash\mathbb{H}$, can be transformed to each other by the action of the Atkin-Lehner involutions on $\Gamma_0(N)\backslash\mathbb{H}$ (Proposition 4.7).

The rest of the paper is organized as follows. In Sect.3, we prove general formulae for $\delta(A)$ and $\widetilde{\delta}(A)$. Here Shioda's Torelli theorem for Abelian surfaces [12] and the technique of discriminant form developed by Nikulin [11] are applied. These weak formulae will be analyzed in more detail for each Picard number. The case of Picard number 2 is well-known, and can also be derived immediately from the weak formula. The case of Picard number 3 is treated in Sect.4, and the case of Picard number 4 is studied in Sect.5. In Sect.6 we will calculate the order of the isometry group $O(D_L)$ for a rank 2 even lattice L. This part is purely algebraic and may be read independently.

2 Conventions

Let A be an Abelian surface. The Néron-Severi (resp. transcendental) lattice of A is denoted by NS_A (resp. T_A). The Picard number of A is denoted by $\rho(A)$. For a curve $C \subset A$ its class in NS_A is written as $[C]$. The *positive cone* $\mathcal{C}_A^+$ of A is the connected component of the open set $\{x \in NS_A \otimes \mathbb{R},\ (x,x) > 0\}$ containing ample classes. The group of Hodge isometries of T_A is denoted by $O_{Hodge}(T_A)$.

Let L be an even lattice, i.e., a free $\mathbb{Z}$-module of finite rank equipped with a non-degenerate integral symmetric bilinear form $(,)$ satisfying $(l,l) \in 2\mathbb{Z}$ for all $l \in L$. The isometry group of L is denoted by $O(L)$. Let $SO(L) := \{\gamma \in O(L),\ \det(\gamma) = 1\}$, which is of index at most 2 in $O(L)$. For an integer $n \in \mathbb{Z}$, $L(n)$ denotes the lattice $(L, n(,))$. An even lattice L is *primitive* if $L \not\simeq L'(n)$ for any even lattice L' and $n > 1$. On the other hand, a sublattice M (resp. a vector l) of L is said to be *primitive* if L/M (resp. $L/\mathbb{Z}l$) is free. In the rest of the paper, the distinction between these two notions of primitivity will be clear from the context.

Let $L^\vee = \mathrm{Hom}(L,\mathbb{Z})$ be the dual lattice of L, which is canonically embedded in the quadratic space $L \otimes \mathbb{Q}$ and contains L. On the finite Abelian group $D_L = L^\vee/L$ a natural quadratic form $q_L : D_L \to \mathbb{Q}/2\mathbb{Z}$ is defined by $q_L(x+L, x+L) = (x,x) + 2\mathbb{Z}$. This finite quadratic form (D_L, q_L), often abbreviated as D_L, is called the *discriminant form* of L. A homomorphism $r_L : O(L) \to O(D_L, q_L)$ is defined naturally. For a primitive sublattice L of an even unimodular lattice M with the orthogonal complement $L^\perp$, there exists a canonical isometry (cf. [11])

$$(D_L, q_L) \xrightarrow{\simeq} (D_{L^\perp}, -q_{L^\perp}). \tag{2.1}$$

Two even lattices L and M are *isogenus* if $L \otimes \mathbb{Z}_p \simeq M \otimes \mathbb{Z}_p$ for every prime number p and $\mathrm{sign}(L) = \mathrm{sign}(M)$. By Nikulin's theorem [11] L and M are isogenus if and only if $(D_L, q_L) \simeq (D_M, q_M)$ and $\mathrm{sign}(L) = \mathrm{sign}(M)$. The *genus* $\mathcal{G}(L)$ of L is the set of isometry classes of even lattices isogenus to L. On the other hand, the set of orientation-preserving isometry classes of *oriented* even lattices isogenus to L is denoted by $\widetilde{\mathcal{G}}(L)$ and called the *proper genus* of L. Writing

$$\mathcal{G}_1(L) := \{M \in \mathcal{G}(L) \mid O(M) \neq SO(M)\} \tag{2.2}$$

and $\mathcal{G}_2(L) := \mathcal{G}(L) - \mathcal{G}_1(L)$, we have

$$|\widetilde{\mathcal{G}}(L)| = |\mathcal{G}_1(L)| + 2|\mathcal{G}_2(L)|. \tag{2.3}$$

The *hyperbolic plane* is the rank 2 even unimodular lattice

$$U = \mathbb{Z}e + \mathbb{Z}f, \quad (e,e) = (f,f) = 0,\ (e,f) = 1. \tag{2.4}$$

Throughout the paper we fix this basis $\{e,f\}$ for U. The orientation-reversing isometry

$$\iota_0 : U \longrightarrow U, \quad \iota_0(e) = f,\ \iota_0(f) = e \tag{2.5}$$

will be used several times.

3 Weak formulae

3.1 A formula for $\delta(A)$

Let A be an Abelian surface. For a decomposition (E_1, E_2) of A we define an embedding $\varphi : U \hookrightarrow NS_A$ by

$$\varphi(e) = [E_1], \quad \varphi(f) = [E_2], \tag{3.1}$$

where $e, f \in U$ are as defined in (2.4). Since $([E_i], [E_i]) = 0$ and $([E_1], [E_2]) = 1$, φ is certainly an embedding of U.

Definition 3.1. Let

$$\Gamma_A := r_{NS}^{-1}\Big(\lambda \circ r_T(O_{Hodge}(T_A))\Big) \subset O(NS_A), \tag{3.2}$$

where $r_{NS} : O(NS_A) \to O(D_{NS_A})$ and $r_T : O(T_A) \to O(D_{T_A})$ are the natural homomorphisms, and $\lambda : O(D_{T_A}) \simeq O(D_{NS_A})$ is the isomorphism induced by the isometry $(D_{T_A}, q_{T_A}) \simeq (D_{NS_A}, -q_{NS_A})$ (see (2.1)).

There is an obvious inclusion $\mathrm{Ker}(r_{NS}) \cdot \{\pm \mathrm{id}\} \subset \Gamma_A$. Let $O_{Hodge}(H^2(A, \mathbb{Z}))$ be the group of Hodge isometries of $H^2(A, \mathbb{Z})$. By Nikulin's theorem ([11] Corollary 1.5.2), the group Γ_A can be written as

$$\Gamma_A = \mathrm{Image}\,(\, O_{Hodge}(H^2(A, \mathbb{Z})) \longrightarrow O(NS_A)\,). \tag{3.3}$$

Proposition 3.2. *Let (E_1, E_2) and (F_1, F_2) be decompositions of an Abelian surface A and let φ, ψ be the corresponding embeddings of U. Then (E_1, E_2) and (F_1, F_2) are isomorphic if and only if $\varphi \in \Gamma_A \cdot \psi$.*

Proof. If (E_1, E_2) and (F_1, F_2) are strictly isomorphic, there exists an automorphism f of A satisfying $f(E_i) = F_i$. Then $f^*([F_i]) = [E_i]$ and $f^*|_{NS_A} \in \Gamma_A$ so that we have $\varphi \in \Gamma_A \cdot \psi$. Let ι_0 be the isometry of U defined in (2.5). The embedding associated to the decomposition (E_2, E_1) is

$$\varphi \circ \iota_0 = \big((\varphi \circ \iota_0 \circ \varphi^{-1})|_{\varphi(U)} \oplus \mathrm{id}_{\varphi(U)^\perp}\big) \circ \varphi \ \in \Gamma_A \cdot \varphi.$$

Therefore φ and ψ are Γ_A-equivalent if (E_1, E_2) and (F_1, F_2) are isomorphic.

Conversely, suppose that $\varphi = \gamma \circ \psi$ for some isometry $\gamma \in \Gamma_A$. By (3.3) γ can be extended to a Hodge isometry $\Phi : H^2(A, \mathbb{Z}) \to H^2(A, \mathbb{Z})$. When $\det(\Phi) = 1$, Shioda's Torelli theorem ([12] Theorem 1) assures the existence of an automorphism f of A such that $f^* = \Phi$ or $-\Phi$. Since Φ preserves the cone $\mathcal{C}_A^+$, we have $f^* = \Phi$. Then $f^*([F_i]) = [E_i]$ so that $f(E_i) = F_i$. On the other hand, when $\det(\Phi) = -1$, consider the Hodge isometry

$$\Psi := \Big((\varphi \circ \iota_0 \circ \varphi^{-1})|_{\varphi(U)} \oplus \mathrm{id}_{\varphi(U)^\perp}\Big) \circ \Phi : H^2(A, \mathbb{Z}) \to H^2(A, \mathbb{Z}).$$

As above, there exists an automorphism g of A such that $g^* = \Psi$. Hence (E_1, E_2) is strictly isomorphic to (F_2, F_1). $\square$

Let

$$\mathrm{Emb}(U, NS_A)$$

be the set of embeddings of U into NS_A. By Proposition 3.2 an injective map

$$\mathrm{Dec}(A) \hookrightarrow \Gamma_A \backslash \mathrm{Emb}(U, NS_A) \tag{3.4}$$

is defined. To prove its surjectivity, we need the following well-known proposition. A non-zero vector $l \in NS_A$ is *isotropic* if $(l, l) = 0$.

Lemma 3.3 (cf.[2])**.** *Every primitive isotropic vector of NS_A contained in the closure of the cone $\mathcal{C}_A^+$ is the class of an elliptic curve in A.*

Now we have

Proposition 3.4. *The map defined in (3.4) is bijective.*

Proof. It suffices to prove the surjectivity. Let us given an embedding $\varphi : U \hookrightarrow NS_A$. Composing with $-\mathrm{id}$ if necessary, we may assume that the vector $\varphi(e)$ is contained in the closure of $\mathcal{C}_A^+$. As $(\varphi(e), \varphi(f)) = 1$, the vector $\varphi(f)$ is also contained in the closure of $\mathcal{C}_A^+$. By the above Lemma 3.3, there exist elliptic curves E_1, E_2 in A such that $[E_1] = \varphi(e)$ and $[E_2] = \varphi(f)$. As $([E_1], [E_2]) = 1$, we have $E_1 \cap E_2 = \{0\}$ so that (E_1, E_2) is a decomposition of A. The embedding associated with (E_1, E_2) is φ. □

Corollary 3.5. *An Abelian surface A is decomposable if and only if NS_A admits an embedding of the hyperbolic plane U.*

Proposition 3.6. *Let L be an even lattice satisfying $NS_A \simeq U \oplus L$. Then*

$$\Big| \Gamma_A \backslash \mathrm{Emb}(U, NS_A) \Big| = \sum_{M \in \mathcal{G}(L)} \Big| O_{Hodge}(T_A) \backslash O(D_M) / O(M) \Big|.$$

Proof. For each even lattice $M \in \mathcal{G}(L)$ there exists an embedding $\varphi_M : U \hookrightarrow NS_A$ with $\varphi_M(U)^\perp \simeq M$ by Nikulin-Kneser's uniqueness theorem (Corollary 1.13.3 of [11]). We have the decomposition

$$\begin{aligned} \Gamma_A \backslash \mathrm{Emb}(U, NS_A) &= \bigsqcup_{M \in \mathcal{G}(L)} \Gamma_A \backslash \{\, \varphi : U \hookrightarrow NS_A,\ \varphi(U)^\perp \simeq M \,\} \\ &= \bigsqcup_{M \in \mathcal{G}(L)} \Gamma_A \backslash \left(O(NS_A) \cdot \varphi_M \right) \\ &\simeq \bigsqcup_{M \in \mathcal{G}(L)} \Gamma_A \backslash O(NS_A) / O(M). \end{aligned}$$

We apply the homomorphism $r : O(NS_A) \to O(D_{NS_A})$, which is surjective by Nikulin's theorem ([11] Theorem 1.14.2). Since $\mathrm{Ker}(r) \subset \Gamma_A$, we obtain

$$\begin{aligned} \Gamma_A \backslash O(NS_A) / O(M) &\simeq r(\Gamma_A) \backslash O(D_{NS_A}) / r(O(M)) \\ &\simeq r(O_{Hodge}(T_A)) \backslash O(D_M) / r(O(M)). \end{aligned}$$

□

By Propositions 3.4 and 3.6 we obtain

Proposition 3.7. *Let A be a decomposable Abelian surface and let L be an even lattice satisfying $NS_A \simeq U \oplus L$. Then the decomposition number $\delta(A)$ is given by*

$$\delta(A) = \sum_{M \in \mathcal{G}(L)} \Big| \, O_{Hodge}(T_A) \backslash O(D_M)/O(M) \, \Big|. \tag{3.5}$$

This formula will be analyzed in more detail in the subsequent sections.

Remark 3.8. The decomposition number is related to the number of principal polarizations. For example see Hayashida [5]. Herbert Lange taught the author that Peter Schuster also studied in his thesis the number of decompositions and principal polarizations by using class numbers of Hermitian forms. See [8] and the references therein for more details.

3.2 A formula for $\widetilde{\delta}(A)$

Let A be an Abelian surface with $\rho(A) = 4$. In this case, the transcendental lattice T_A is a rank 2 positive-definite even lattice and the Hodge structure of T_A induces a natural orientation of T_A. An isometry of T_A preserves the Hodge structure if and only if it preserves the orientation. Thus we have

$$O_{Hodge}(T_A) = SO(T_A). \tag{3.6}$$

Since $(D_{T_A}, q_{T_A}) \simeq (D_{NS_A}, -q_{NS_A})$, the lattices NS_A and $U \oplus T_A(-1)$ are isogenus. It follows from Nikulin-Kneser's uniqueness theorem that

$$NS_A \simeq U \oplus T_A(-1). \tag{3.7}$$

In particular, A is always decomposable ([13]). Let

$$S\Gamma_A := \Gamma_A \cap SO(NS_A). \tag{3.8}$$

For a Hodge isometry Φ of $H^2(A, \mathbb{Z})$ we have $\det(\Phi) = 1$ if and only if $\Phi|_{NS_A} \in S\Gamma_A$. With this fact in mind, we can prove the following proposition similarly as Propositions 3.2 and 3.4.

Proposition 3.9. *Suppose that $\rho(A) = 4$. For a decomposition (E_1, E_2) of A we define the embedding $\varphi : U \hookrightarrow NS_A$ by the equation (3.1). Then this assignment induces the bijection*

$$\widetilde{\mathrm{Dec}}(A) \simeq S\Gamma_A \backslash \mathrm{Emb}(U, NS_A).$$

For each lattice $T \in \mathcal{G}(T_A)$ we can find an embedding $\varphi_T : U \hookrightarrow NS_A$ with $\varphi_T(U)^\perp \simeq T(-1)$. Then, as like the proof of Proposition 3.6,

$$S\Gamma_A \backslash \mathrm{Emb}(U, NS_A) = \bigsqcup_{T \in \mathcal{G}(T_A)} S\Gamma_A \backslash (O(NS_A) \cdot \varphi_T).$$

The orbit $O(NS_A)\cdot\varphi_T$ is decomposed as

$$O(NS_A)\cdot\varphi_T = SO(NS_A)\cdot\varphi_T \,\cup\, SO(NS_A)\cdot(\varphi_T\circ\iota_0), \tag{3.9}$$

where ι_0 is the isometry of U defined by the equation (2.5).

Lemma 3.10. *We have $\varphi_T\circ\iota_0\in SO(NS_A)\cdot\varphi_T$ if and only if $SO(T)\neq O(T)$.*

Proof. If $\varphi_T\circ\iota_0=\gamma\circ\varphi_T$ for some $\gamma\in SO(NS_A)$, this γ can be written as

$$\gamma = (\varphi_T\circ\iota_0\circ\varphi_T^{-1})|_{\varphi_T(U)}\oplus\gamma'$$

for some $\gamma'\in O(T(-1))=O(T)$. Then $\det(\gamma')=\det(\gamma)\cdot\det(\iota_0)^{-1}=-1$ so that $SO(T)\neq O(T)$. The converse is proved similarly. $\square$

Therefore we have

$$|\,S\Gamma_A\backslash(O(NS_A)\cdot\varphi_T)\,| = \begin{cases} |\,S\Gamma_A\backslash SO(NS_A)/SO(T)\,|, & \text{if } T\in\mathcal{G}_1(T_A),\\ 2\cdot|\,S\Gamma_A\backslash SO(NS_A)/SO(T)\,|, & \text{if } T\in\mathcal{G}_2(T_A),\end{cases}$$

where $\mathcal{G}_i(T_A)$ are the subsets of $\mathcal{G}(T_A)$ defined in (2.2). Now an imitation of the proof of Proposition 3.6 yields the following formula involving the proper genus $\widetilde{\mathcal{G}}(T_A)$.

Proposition 3.11. *Let A be an Abelian surface with $\rho(A)=4$. Then*

$$\widetilde{\delta}(A)=\sum_{T\in\widetilde{\mathcal{G}}(T_A)}\Big|\,SO(T_A)\backslash O(D_{T_A})/SO(T)\,\Big|.$$

We will study this formula more closely in Section 5.

4 The case of Picard number 3

4.1 Counting formula

Let A be a decomposable Abelian surface with $\rho(A)=3$. Then $NS_A\simeq U\oplus\langle -2N\rangle$ for some $N\in\mathbb{Z}_{>0}$. This natural number N may be calculated by

$$N=\frac{1}{2}\det(NS_A)=-\frac{1}{2}\det(T_A). \tag{4.1}$$

We also have the following.

Proposition 4.1. *Let (E_1,E_2) be a decomposition of A. Then*

$$N=\min\Big\{\deg\phi\;\Big|\;\phi: E_1\to E_2 \text{ isogeny}\Big\}. \tag{4.2}$$

In particular, the right hand side of (4.2) is independent of the choice of (E_1,E_2).

Proof. Let $l \in NS_A$ be a generator of the rank 1 lattice

$$(\mathbb{Z}[E_1] + \mathbb{Z}[E_2])^{\perp} \cap NS_A \simeq \langle -2N \rangle.$$

The class $[E_1] + l + N[E_2] \in NS_A$ is a primitive isotropic vector contained in the closure of $\mathcal{C}_A^+$ so that there exists an elliptic curve E in A with $[E] = [E_1] + l + N[E_2]$. Since $([E], [E_2]) = 1$ (resp. $([E], [E_1]) = N$), the degree of the projection $E \to E_1$ (resp. $E \to E_2$) is 1 (resp. N). Thus we obtain an isogeny $E_1 \to E \to E_2$ of degree N.

Conversely, let $\phi : E_1 \to E_2$ be an arbitrary isogeny. Its graph $\Gamma \subset A$ is an elliptic curve satisfying $([\Gamma], [E_2]) = 1$ and $([\Gamma], [E_1]) = \deg\phi$. We can write $[\Gamma] = [E_1] + al + (\deg\phi)[E_2]$ for some $a \in \mathbb{Z}$. Then we have $\deg\phi = a^2 N \geq N$. $\square$

It follows that

$$\widetilde{\delta}(A) = \begin{cases} \delta(A), & \text{if } N = 1, \\ 2\delta(A), & \text{if } N > 1. \end{cases}$$

Proposition 4.2 (cf. [5])**.** *Let A be a decomposable Abelian surface with $\rho(A) = 3$ and $\det(T_A) = -2N$. Then $\delta(A) = 2^{\tau(N)-1}$. We also have*

$$\widetilde{\delta}(A) = \begin{cases} 1, & \text{if } N = 1, \\ 2^{\tau(N)}, & \text{if } N > 1. \end{cases}$$

Proof. The right hand side of the formula (3.5) can be written as

$$\left| O_{Hodge}(T_A) \backslash O(D_{\langle -2N \rangle}) / O(\langle -2N \rangle) \right|.$$

As $\mathrm{rk}(T_A) = 3$ is odd, it follows from Appendix B of [6] that $O_{Hodge}(T_A) = \{\pm \mathrm{id}\}$. The isometry group $O(\langle -2N \rangle)$ is clearly $\{\pm \mathrm{id}\}$. Since $D_{\langle -2N \rangle} = \langle \frac{-1}{2N} \rangle \simeq \mathbb{Z}/2N\mathbb{Z}$, we have (cf. [6])

$$\left| O(D_{\langle -2N \rangle}) \right| = \begin{cases} 1, & N = 1, \\ 2^{\tau(N)}, & N > 1. \end{cases}$$

$\square$

Note that $\widetilde{\delta}(A)$ can be represented simply as

$$\widetilde{\delta}(A) = |O(D_{NS_A})| = |O(D_{T_A})|.$$

Proposition 4.2 was first proved by Hayashida [5]. Hayashida defined the number N as the minimal degree of isogeny $E \to F$, where (E, F) is a decomposition of A.

4.2 Construction of decompositions

Let $A = E_1 \times E_2$ be a decomposable Abelian surface with $\rho(A) = 3$. We shall construct representatives of $\mathrm{Dec}(A)$ from (E_1, E_2). Let $e := [E_1]$, $f := [E_2]$, and l be a generator of the lattice $\langle e, f\rangle^{\perp} \cap NS_A$. Firstly we construct representatives of the quotient set $\Gamma_A \backslash \mathrm{Emb}(U, NS_A)$ as follows. Let

$$\Sigma := \left\{ (r_\sigma, s_\sigma) \;\middle|\; r_\sigma, s_\sigma \in \mathbb{Z}_{>0},\ (r_\sigma, s_\sigma) = 1,\ r_\sigma s_\sigma = N,\ r_\sigma \leq s_\sigma \right\}. \tag{4.3}$$

We have $|\Sigma| = 2^{\tau(n)-1}$. For each $\sigma \in \Sigma$ choose integers $a_\sigma, b_\sigma \in \mathbb{Z}$ satisfying $a_\sigma r_\sigma + b_\sigma s_\sigma = 1$ and put

$$\begin{aligned} e_\sigma &:= r_\sigma e + s_\sigma f + l, && (4.4)\\ f_\sigma &:= b_\sigma^2 s_\sigma e + a_\sigma^2 r_\sigma f - a_\sigma b_\sigma l, && (4.5)\\ l_\sigma &:= 2N b_\sigma e - 2N a_\sigma f + (b_\sigma s_\sigma - a_\sigma r_\sigma) l. && (4.6) \end{aligned}$$

The vectors e_σ, f_σ define an embedding $\varphi_\sigma : U \hookrightarrow NS_A$ with $\varphi_\sigma(U)^{\perp} = \mathbb{Z} l_\sigma$.

Lemma 4.3. *The set $\{\varphi_\sigma\}_{\sigma \in \Sigma}$ of embeddings represents $\Gamma_A \backslash \mathrm{Emb}(U, NS_A)$ completely.*

Proof. As $|\Sigma| = |\,\Gamma_A \backslash \mathrm{Emb}(U, NS_A)\,| = 2^{\tau(N)-1}$, it suffices to show that $\varphi_\sigma \notin \Gamma_A \cdot \varphi_{\sigma'}$ if $\sigma \neq \sigma'$. There are exactly two isometries of NS_A, say $\gamma^{+}_{\sigma',\sigma}$ and $\gamma^{-}_{\sigma',\sigma}$, satisfying $\gamma^{\pm}_{\sigma',\sigma} \circ \varphi_\sigma = \varphi_{\sigma'}$:

$$\gamma^{\pm}_{\sigma',\sigma}(e_\sigma) = e_{\sigma'}, \quad \gamma^{\pm}_{\sigma',\sigma}(f_\sigma) = f_{\sigma'}, \quad \gamma^{\pm}_{\sigma',\sigma}(l_\sigma) = \pm l_{\sigma'}.$$

When N is odd, $D_{NS_A} \simeq \mathbb{Z}/2N\mathbb{Z}$ is decomposed as

$$D_{NS_A} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{i=1}^{\tau(N)} \mathbb{Z}/p_i^{e_i}\mathbb{Z}, \tag{4.7}$$

where $N = \prod p_i^{e_i}$ is the prime decomposition of N. We write

$$\frac{l}{2N} = x_0 + x_1 + \cdots + x_{\tau(N)} \ \in D_{NS_A}$$

with respect to this decomposition (4.7). Direct calculations show that

$$\frac{l_\sigma}{2N} \equiv (b_\sigma s_\sigma - a_\sigma r_\sigma)\frac{l}{2N} = x_0 + \Big(\sum_{p_i | r_\sigma} x_i\Big) - \Big(\sum_{p_j | s_\sigma} x_j\Big).$$

Now, if $r(\gamma^{\pm}_{\sigma',\sigma}) \in \{\pm \mathrm{id}\} \subset O(D_{NS_A})$, we must have $r_\sigma = r_{\sigma'}$ or $r_\sigma = s_{\sigma'}$, which implies that $\sigma = \sigma'$ by the definition of Σ. Thus we have $\sigma = \sigma'$ if $\varphi_{\sigma'} \in \Gamma_A \cdot \varphi_\sigma$. The argument if N is even is similar. $\square$

Next we find decompositions corresponding to the embeddings $\{\varphi_\sigma\}$. Let $\phi : E_1 \to E_2$ be an isogeny of degree N, the existence of which is guaranteed by Proposition 4.1. Its kernel $G := \mathrm{Ker}(\phi) \subset E_1$ is a cyclic group of order N and is uniquely determined by the ordered pair (E_1, E_2). Let

$$G = \bigoplus_{i=1}^{\tau(N)} G_i, \quad |G_i| = p_i^{e_i}, \tag{4.8}$$

be the decomposition of G into p-groups and put

$$G_{\sigma,1} := \bigoplus_{p_i | r_\sigma} G_i, \quad G_{\sigma,2} := \bigoplus_{p_j | s_\sigma} G_j. \tag{4.9}$$

We have a canonical decomposition $G = G_{\sigma,1} \oplus G_{\sigma,2}$. If we denote

$$E_{\sigma,i} := E_1/G_{\sigma,i}, \quad i = 1, 2, \tag{4.10}$$

then the isogeny $\phi : E_1 \to E_2$ can be factorized as

$$E_1 \xrightarrow{\phi^+_{\sigma,i}} E_{\sigma,i} \xrightarrow{\phi^-_{\sigma,i}} E_2, \quad i = 1, 2 \tag{4.11}$$

with

$$\deg(\phi^+_{\sigma,1}) = \deg(\phi^-_{\sigma,2}) = r_\sigma, \quad \deg(\phi^+_{\sigma,2}) = \deg(\phi^-_{\sigma,1}) = s_\sigma.$$

Let $\widehat{\phi^\pm_{\sigma,i}}$ be the dual isogeny of $\phi^\pm_{\sigma,i}$.

Lemma 4.4. *We have* $\phi^+_{\sigma,2} \circ \widehat{\phi^+_{\sigma,1}} = \widehat{\phi^-_{\sigma,2}} \circ \phi^-_{\sigma,1} : E_{\sigma,1} \to E_{\sigma,2}$.

Proof. Let

$$\varphi := \phi^+_{\sigma,2} \circ \widehat{\phi^+_{\sigma,1}} - \widehat{\phi^-_{\sigma,2}} \circ \phi^-_{\sigma,1} : E_{\sigma,1} \longrightarrow E_{\sigma,2}.$$

Since

$$\phi^-_{\sigma,2} \circ \varphi = \phi^-_{\sigma,1} \circ \phi^+_{\sigma,1} \circ \widehat{\phi^+_{\sigma,1}} - r_\sigma \phi^-_{\sigma,1} = \phi^-_{\sigma,1} r_\sigma - r_\sigma \phi^-_{\sigma,1} = 0,$$

then we have $\varphi(E_{\sigma,1}) \subset \mathrm{Ker}(\phi^-_{\sigma,2})$. By the discreteness of $\mathrm{Ker}(\phi^-_{\sigma,2})$ we conclude that $\varphi(E_{\sigma,1}) = \{0\}$. □

Compared with the factorizations (4.11), Lemma 4.4 represents a symmetry between the pairs (E_1, E_2) and $(E_{\sigma,1}, E_{\sigma,2})$.

Proposition 4.5. *The homomorphism*

$$\alpha_\sigma := \begin{pmatrix} \widehat{\phi^+_{\sigma,1}} & b_\sigma \widehat{\phi^+_{\sigma,2}} \\ -\phi^-_{\sigma,1} & a_\sigma \phi^-_{\sigma,2} \end{pmatrix} : E_{\sigma,1} \times E_{\sigma,2} \longrightarrow E_1 \times E_2 \tag{4.12}$$

is an isomorphism. In other words, the pair $(\alpha_\sigma(E_{\sigma,1}), \alpha_\sigma(E_{\sigma,2}))$ *gives a decomposition of* A.

Proof. We put

$$\beta_\sigma := \begin{pmatrix} a_\sigma \phi^+_{\sigma,1} & -b_\sigma \widehat{\phi^-_{\sigma,1}} \\ \phi^+_{\sigma,2} & \widehat{\phi^-_{\sigma,2}} \end{pmatrix} : E_1 \times E_2 \longrightarrow E_{\sigma,1} \times E_{\sigma,2}.$$

With the aid of Lemma 4.4 we can show that $\beta_\sigma \circ \alpha_\sigma = \mathrm{id}$ and $\alpha_\sigma \circ \beta_\sigma = \mathrm{id}$. □

Theorem 4.6. *Let $A = E_1 \times E_2$ be a decomposable Abelian surface with $\rho(A) = 3$. Then the decompositions $\{(\alpha_\sigma(E_{\sigma,1}), \alpha_\sigma(E_{\sigma,2}))\}_{\sigma\in\Sigma}$ of A defined by (4.10) and (4.12) represent* $\mathrm{Dec}(A)$ *completely.*

Proof. We may assume that $N > 1$. Let $C_{\sigma,i} := \alpha_\sigma(E_{\sigma,i}) \subset A$. By calculating the degrees of the projections $C_{\sigma,i} \to E_j$, we see that

$$([C_{\sigma,1}], [C_{\sigma,2}]) = (e_\sigma, f_\sigma) \quad \text{or} \quad (e_\sigma - 2l, f_\sigma + 2a_\sigma b_\sigma l),$$

where e_σ, f_σ are the vectors defined by (4.4) and (4.5). If $([C_{\sigma,1}], [C_{\sigma,2}]) = (e_\sigma, f_\sigma)$, the embedding of U associated to the decomposition $(C_{\sigma,1}, C_{\sigma,2})$ is φ_σ. If $([C_{\sigma,1}], [C_{\sigma,2}]) = (e_\sigma - 2l, f_\sigma + 2a_\sigma b_\sigma l)$, the corresponding embedding is

$$\left(\mathrm{id}_{\langle e,f\rangle} \oplus -\mathrm{id}_{\langle l\rangle}\right) \circ \varphi_\sigma \quad \in \Gamma_A \cdot \varphi_\sigma.$$

Thus our claim follows from Lemma 4.3. □

The construction of the decompositions $(E_{\sigma,1}, E_{\sigma,2})$ is related with the geometries of elliptic modular curves. Let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),\ c \equiv 0 \mod N \right\}.$$

As is well-known, the congruence modular curve $\Gamma_0(N)\backslash\mathbb{H}$ is the moduli space of elliptic curves with cyclic subgroups of order N. To a decomposition (E, F) of A we associate a point in $\Gamma_0(N)\backslash\mathbb{H}$ by considering the the pair $(E, \mathrm{Ker}(\phi))$, where $\phi : E \to F$ is an isogeny of the minimal degree N. Let us denote this point by

$$\omega(E, F) \ \in \Gamma_0(N)\backslash\mathbb{H}.$$

When $N > 1$, the Abelian group $G = (\mathbb{Z}/2\mathbb{Z})^{\tau(N)}$ acts on the curve $\Gamma_0(N)\backslash\mathbb{H}$ by the Atkin-Lehner involutions ([9], see also [7]). A comparison of the definition of $(E_{\sigma,1}, E_{\sigma,2})$ and that of Atkin-Lehner involutions yields the following.

Proposition 4.7. *Let $A = E_1 \times E_2$ be a decomposable Abelian surface with $\rho(A) = 3$. Assume that $E_1 \not\cong E_2$. Then we have*

$$G \cdot \omega(E_1, E_2) = \{\, \omega(E_{\sigma,1}, E_{\sigma,2}),\ \omega(E_{\sigma,2}, E_{\sigma,1})\,\}_{\sigma\in\Sigma}.$$

In other words, all members of $\widetilde{\mathrm{Dec}}(A)$ can be constructed from a given (E_1, E_2) by the action of the Atkin-Lehner involutions.

Remark 4.8. Before finishing this section, let us indicate a way of generalizing Proposition 4.2 to a counting formula for the set

$$\mathrm{El}(A) := \mathrm{Aut}(A)\backslash \{\, E \subset A \text{ elliptic curve} \,\}.$$

If we denote

$$I(NS_A) := \{\, \mathbb{Z}l \subset NS_A, \text{ primitive isotropic sublattice of rank1} \,\},$$

then $\mathrm{El}(A)$ is naturally identified with the quotient set $\mathrm{Aut}(A)\backslash I(NS_A)$. Let

$$\mathrm{Aut}(A)_0 = \{\, f \in \mathrm{Aut}(A),\ \ f^*|_{T_A} = \mathrm{id}_{T_A} \,\}$$

be the group of symplectic automorphisms of A, which is of index 2 in $\mathrm{Aut}(A)$. Firstly we study the set $\mathrm{Aut}(A)_0\backslash I(NS_A)$. The image of $\mathrm{Aut}(A)_0$ in $O(NS_A)$ is equal to the group

$$SO(NS_A)_0^+ := \{\, \gamma \in SO(NS_A) \mid\ r_{NS}(\gamma) = \mathrm{id} \in O(D_{NS_A}),\ \gamma(\mathcal{C}_A^+) = \mathcal{C}_A^+ \,\}.$$

By the theory of Baily-Borel compactification [1], the set $SO(NS_A)_0^+\backslash I(NS_A)$ is canonically identified with the set of cusps of the modular variety associated to the orthogonal group $SO(NS_A)_0^+$. Via the Clifford algebra construction for example, the group $SO(NS_A)_0^+$ is shown to be isomorphic to the congruence modular group $\Gamma_0(N)$, and the isomorphism of groups induces that of the corresponding modular curves. Hence the set $\mathrm{Aut}(A)_0\backslash I(NS_A)$ is identified with the set of $\Gamma_0(N)$-cusps, which is well-known. Now the number $|\mathrm{El}(A)|$ is calculated by looking the action of the group $\mathrm{Aut}(A)/\mathrm{Aut}(A)_0 \simeq \mathbb{Z}/2\mathbb{Z}$ on the set of $\Gamma_0(N)$-cusps. This involution is calculated to be $r \mapsto -r$ for $r \in \mathbb{Q}$.

5 The case of Picard number 4

5.1 Counting formulae

Let A be an Abelian surface with $\rho(A) = 4$. As is well-known, the isomorphism class of A is uniquely determined by the proper equivalence class of the transcendental lattice T_A [13]. Therefore it seems natural to express $\delta(A)$ and $\widetilde{\delta}(A)$ in terms of the arithmetic of T_A. Since T_A is positive-definite of rank 2, the group $SO(T_A)$ is described completely as

$$SO(T_A) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } T_A \not\simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}, \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}, \\ \mathbb{Z}/4\mathbb{Z}, & \text{if } T_A \simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}, \\ \mathbb{Z}/6\mathbb{Z}, & \text{if } T_A \simeq \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}. \end{cases}$$

First we consider the general case : $T_A \not\simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}, \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}$. The group $SO(T_A)$ consists of $\{\pm\mathrm{id}\}$ and the genus $\mathcal{G}(T_A)$ does not contain the lattices $\begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}, \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}$ because they are unique in their genera.

Proposition 5.1. *Suppose that* $\rho(A) = 4$ *and* $T_A \not\simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}, \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}$. *Then*

$$\delta(A) = \sum_{T \in \mathcal{G}(T_A)} |\, O(D_T)/O(T)\,|.$$

$$\widetilde{\delta}(A) = \frac{1}{2} \cdot |\, \widetilde{\mathcal{G}}(T_A)\,| \cdot |\, O(D_{T_A})\,|.$$

$$\delta_0(A) = \begin{cases} \frac{1}{2} \cdot |\, O(D_{T_A})\,|, & \textit{if } \begin{pmatrix} 2 & 1 \\ 1 & 2c \end{pmatrix} \textit{ or } \begin{pmatrix} 2 & 0 \\ 0 & 2c \end{pmatrix} \in \mathcal{G}(T_A), \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The first two equalities are deduced immediately from Propositions 3.7 and 3.11. Note that $\mathrm{id} \neq -\mathrm{id}$ in $O(D_{T_A})$ because $|D_{T_A}| > 4$. For the third equality, we have

$$\begin{aligned} \delta_0(A) &= 2\delta(A) - \widetilde{\delta}(A) \\ &= \sum_{T \in \mathcal{G}_1(T_A)} \Big\{ 2 \cdot |O(D_T)/O(T)| \; - \; |O(D_T)/\{\pm \mathrm{id}\}| \Big\} \\ &= |\, \mathcal{G}_0(T_A)\,| \cdot |\, O(D_{T_A})/\{\pm \mathrm{id}\}\,|, \end{aligned}$$

where $\mathcal{G}_0(T_A) := \{\, T \in \mathcal{G}_1(T_A) \mid r_T(O(T)) = \{\pm \mathrm{id}\}\,\}$. Let $T \in \mathcal{G}_0(T_A)$. Since $SO(T) \neq O(T)$, it follows from the classification of ambiguous binary forms ([3], Chapter 14.4) that T is isometric to one of the following lattices :

$$\begin{pmatrix} 2a & 0 \\ 0 & 2c \end{pmatrix}, \quad \begin{pmatrix} 2a & a \\ a & 2c \end{pmatrix}, \quad a, c \in \mathbb{Z}_{>0}.$$

If $T = \begin{pmatrix} 2a & 0 \\ 0 & 2c \end{pmatrix}$, T has the orientation-reversing isometry $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By the requirement that $r_T(\gamma) \in \{\pm \mathrm{id}\}$, either a or c must be equal to 1. When $T = \begin{pmatrix} 2a & a \\ a & 2c \end{pmatrix}$, T admits the orientation-reversing isometry $\sigma = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$. Similarly a or $4c-a$ must be equal to 1. In both cases T is isometric to $\begin{pmatrix} 2 & 1 \\ 1 & 2c \end{pmatrix}$. In conclusion, $\mathcal{G}_0(T_A)$ is either empty or consists of only one class of the type $\begin{pmatrix} 2 & 0 \\ 0 & 2c \end{pmatrix}$ or $\begin{pmatrix} 2 & 1 \\ 1 & 2c \end{pmatrix}$, $c > 0$. □

Next we study the case $T_A \simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}$. The group $SO(T_A)$ is the cyclic group $\mathbb{Z}/4\mathbb{Z}$ generated by the rotation $\gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and the group $O(T_A)$ is the dihedral group of order 8 generated by γ_1 and the reflection $\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with the relations $\gamma_1^4 = \gamma_2^2 = (\gamma_1\gamma_2)^2 = 1$. The genus $\mathcal{G}(T_A) = \widetilde{\mathcal{G}}(T_A)$ consists

only of T_A. The discriminant form D_{T_A} is the group $(\mathbb{Z}/2n\mathbb{Z})^2$ endowed with the quadratic form $\begin{pmatrix} (2n)^{-1} & 0 \\ 0 & (2n)^{-1} \end{pmatrix}$. For brevity, the image of γ_1 in $O(D_{T_A})$ is again denoted by γ_1. Straight calculations yield

Lemma 5.2. *Let T_A and γ_1 be as above and assume that $n > 1$. Then the homomorphism $O(T_A) \to O(D_{T_A})$ is injective. For an isometry $\gamma \in O(D_{T_A})$ we have $\gamma^{-1}\gamma_1\gamma = \det(\gamma)\gamma_1$. As a result, we have*

$$| \langle \gamma_1 \rangle \cdot \gamma \cdot \langle \gamma_1 \rangle | = \begin{cases} 4, & \det(\gamma) = \pm 1, \\ 8, & \det(\gamma) \neq \pm 1. \end{cases}$$

When $n > 1$, considering the determinant $\det(\gamma) \in \mathbb{Z}/2n\mathbb{Z}$ for $\gamma \in O(D_{T_A})$ induces a surjective homomorphism

$$O(D_{T_A}) \twoheadrightarrow (\mathbb{Z}/2\mathbb{Z})^{\tau(n)}. \tag{5.1}$$

Then we have

Proposition 5.3. *Assume that $\rho(A) = 4$ and $T_A \simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}$. Then*

$$\delta(A) = \begin{cases} 1, & n = 1, \\ (2^{-4} + 2^{-\tau(n)-3}) \cdot | O(D_{T_A}) |, & n > 1, \end{cases}$$

and

$$\widetilde{\delta}(A) = \begin{cases} 1, & n = 1, \\ 2\delta(A), & n > 1. \end{cases}$$

Proof. We may assume that $n > 1$. It follows from Proposition 3.11 and Lemma 5.2 that

$$\widetilde{\delta}(A) = | SO(T_A) \backslash O(D_{T_A}) / SO(T_A) | = |O(D_{T_A})| \cdot \frac{2^{\tau(n)-1} + 1}{2^{\tau(n)-1}} \cdot \frac{1}{8}$$

The formula for $\delta(A)$ is derived similarly. □

Finally we consider the case $T_A \simeq \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}$. The group $SO(T_A)$ is the cyclic group $\mathbb{Z}/6\mathbb{Z}$ generated by the rotation $\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, and the group $O(T_A)$ is the dihedral group of order 12 generated by σ_1 and the reflection $\sigma_2 = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ with the relations $\sigma_1^6 = \sigma_2^2 = (\sigma_1\sigma_2)^2 = 1$. The image of σ_1 in $O(D_{T_A})$ is again denoted by σ_1. The genus $\mathcal{G}(T_A) = \widetilde{\mathcal{G}}(T_A)$ consists only of T_A.

Lemma 5.4. *If a matrix $\gamma \in GL_2(\mathbb{Z}/n\mathbb{Z})$ satisfies $\det(\gamma)^2 = 1$ and*

$${}^t\gamma \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \gamma \equiv \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mod \begin{pmatrix} 2n\mathbb{Z} & n\mathbb{Z} \\ n\mathbb{Z} & 2n\mathbb{Z} \end{pmatrix},$$

then we have

$$\gamma^{-1}\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}\gamma = \det(\gamma)\cdot\begin{pmatrix}\frac{\det(\gamma)-1}{2} & -1\\ 1 & \frac{\det(\gamma)+1}{2}\end{pmatrix}. \tag{5.2}$$

Proof. This lemma is proved by direct calculation. □

The discriminant group D_{T_A} contains the subgroup $n^{-1}T_A/T_A$ of index 3. We consider the quadratic form on $n^{-1}T_A/T_A$ induced from the discriminant form. The basis of T_A induces that of $n^{-1}T_A/T_A$, with respect to which the quadratic form is written as $\begin{pmatrix}2n^{-1} & n^{-1}\\ n^{-1} & 2n^{-1}\end{pmatrix}$. As the subgroup $n^{-1}T_A/T_A$ coincides with the subgroup $\{x \in D_{T_A} | nx = 0\}$, it is preserved by the action of $O(D_{T_A})$ so that we have a natural homomorphism

$$\varphi : O(D_{T_A}) \to O(n^{-1}T_A/T_A).$$

With respect to the basis of $n^{-1}T_A/T_A$, the isometry $\varphi(\sigma_1)$ is represented as $\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$.

Lemma 5.5. *Let T_A, σ_1, and φ be as above and assume that $n > 1$. Then the homomorphism $O(T_A) \to O(D_{T_A})$ is injective. For $\gamma \in O(D_{T_A})$ we have*

$$|\,\langle\sigma_1\rangle\cdot\gamma\cdot\langle\sigma_1\rangle\,| = \begin{cases} 6, & \det(\varphi(\gamma)) = \pm 1,\\ 18, & \det(\varphi(\gamma)) \neq \pm 1.\end{cases}$$

Proof. The first assertion is proved immediately. We prove the second assertion. First consider the case $3|n$. The quadratic form on $n^{-1}T_A/T_A \subset D_{T_A}$ is degenerated. We can show that the natural homomorphism φ is injective and that $\det(\varphi(\gamma))^2 = 1 \in \mathbb{Z}/n\mathbb{Z}$ for $\gamma \in O(D_{T_A})$. By applying Lemma 5.4 to $\varphi(\gamma)$ and $\varphi(\sigma_1)$, we see that

$$|\,\langle\varphi(\sigma_1)\rangle\cdot\varphi(\gamma)\cdot\langle\varphi(\sigma_1)\rangle\,| = \begin{cases} 6, & \det(\varphi(\gamma)) = \pm 1,\\ 18, & \det(\varphi(\gamma)) \neq \pm 1.\end{cases}$$

Next consider the case $3 \nmid n$. By the orthogonal decomposition

$$D_{T_A} \;=\; (nT_A^\vee/T_A) \oplus (n^{-1}T_A/T_A) \;\simeq\; \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})^2,$$

we have a canonical decomposition

$$O(D_{T_A}) \;=\; O(nT_A^\vee/T_A) \oplus O(n^{-1}T_A/T_A), \tag{5.3}$$

so that φ is the natural projection with the kernel $O(nT_A^\vee/T_A) \simeq \mathbb{Z}/2\mathbb{Z}$. When $\det(\varphi(\gamma)) = \pm 1$, it follows from Lemma 5.4 that $\langle\sigma_1\rangle$ is a normal subgroup of $O(D_{T_A})$. When $\det(\varphi(\gamma)) \neq \pm 1$, we have again by Lemma 5.4 that

$$|\langle\sigma_1\rangle\cdot\gamma\cdot\langle\sigma_1\rangle| = |\langle\varphi(\sigma_1)\rangle\cdot\varphi(\gamma)\cdot\langle\varphi(\sigma_1)\rangle| = 18.$$

□

When $n > 2$ is odd (resp. even), considering the determinant $\det(\varphi(\gamma)) \in \mathbb{Z}/n\mathbb{Z}$ for $\gamma \in O(D_{T_A})$ induces a surjective homomorphism

$$O(D_{T_A}) \twoheadrightarrow (\mathbb{Z}/2\mathbb{Z})^{\tau(n)} \quad (\text{resp. } O(D_{T_A}) \twoheadrightarrow (\mathbb{Z}/2\mathbb{Z})^{\tau(2^{-1}n)}).$$

Similarly as Proposition 5.3, we have

Proposition 5.6. *Assume that* $\rho(A) = 4$ *and* $T_A \simeq \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}$. *Then*

$$\delta(A) = \begin{cases} 1, & n = 1, \\ 3^{-2} \cdot (2^{-2} + 2^{-\tau(2^{-1}n)}) \cdot |\, O(D_{T_A})\,|, & n : \textit{even}, \\ 3^{-2} \cdot (2^{-2} + 2^{-\tau(n)}) \cdot |\, O(D_{T_A})\,|, & n : \textit{odd} > 1 \end{cases}$$

and

$$\widetilde{\delta}(A) = \begin{cases} 1, & n = 1, \\ 2\delta(A), & n > 1. \end{cases}$$

5.2 Some conclusions

From Propositions 5.1, 5.3, and 5.6, we have

Corollary 5.7. *Let* A *and* B *be Abelian surfaces with Picard number* 4. *If* NS_A *is isometric to* NS_B, *or equivalently if* T_A *is isogenus to* T_B, *then* $\delta(A) = \delta(B)$ *and* $\widetilde{\delta}(A) = \widetilde{\delta}(B)$.

Corollary 5.8. *Let* A *be an Abelian surface with* $\rho(A) = 4$. *Then we have* $\delta_0(A) \neq 0$ *if and only if* T_A *is primitive and belongs to a principal genus, i.e.,* T_A *is isogenus to either* $\begin{pmatrix} 2 & 0 \\ 0 & 2c \end{pmatrix}$ *or* $\begin{pmatrix} 2 & 1 \\ 1 & 2c \end{pmatrix}$ *for some* $c \in \mathbb{Z}_{>0}$.

Note that Corollary 5.8 can also be proved by Shioda-Mitani's ideal-theoretic method ([13], Section 4). It must be well-known to experts. For an odd prime number p and $a \in (\mathbb{Z}/p\mathbb{Z})^{\times}$, let $\chi_p(a) = \left(\frac{a}{p}\right)$ be the Legendre symbol. For an odd number $n \equiv 1 \mod 4$, we define

$$\chi_2(n) = \begin{cases} 1, & n \equiv 1 \mod 8, \\ -1, & n \equiv 5 \mod 8. \end{cases}$$

By Proposition 6.8 proved in Section 6 independently, we have the following.

Corollary 5.9. *Let* A *be an Abelian surface with* $\rho(A) = 4$ *and assume that* $T_A \not\simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}, \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}$. *For a natural number* $N > 1$ *let* A_N *be the Abelian surface with* T_{A_N} *properly equivalent to the form* $T_A(N)$.

(1) *The number* $\widetilde{\delta}(A_N)$ *is divisible by* $\widetilde{\delta}(A)$.

(2) *If* N *is coprime to* $\det(T_A)$, *then*

$$\widetilde{\delta}(A_N) = \widetilde{\delta}(A) \cdot 2^{\tau(N)} \cdot N \cdot \prod_{p|N} \Big(1 - \frac{\chi_p(-\det(T_A))}{p}\Big). \tag{5.4}$$

(3) *If T_A is primitive and $N|\det(T_A)^a$ for some $a \in \mathbb{Z}_{>0}$, then*

$$\widetilde{\delta}(A_N) = \widetilde{\delta}(A) \cdot 2^{\tau(N)} \cdot N. \tag{5.5}$$

We compare our formula with Shioda-Mitani's formula.

Theorem 5.10 (Shioda-Mitani [13], Theorem 4.7)**.** *Suppose that $\rho(A) = 4$ and that T_A is primitive, i.e, T_A is not isometric to $L(n)$ for any even lattice L and $n > 1$. Let $\mathcal{O}$ be the unique order in an imaginary quadratic field with discriminant $d(\mathcal{O}) = -\det(T_A)$, and $\mathcal{C}(\mathcal{O})$ be the ideal class group of $\mathcal{O}$. Then*

$$\widetilde{\delta}(A) = |\mathcal{C}(\mathcal{O})|. \tag{5.6}$$

Corollary 5.11. *Let A, $\mathcal{O}$ be as in Theorem 5.10. Then*

$$|\mathcal{C}(\mathcal{O})| = \frac{1}{2} \cdot |\, \widetilde{\mathcal{G}}(T_A)\, | \cdot |\, O(D_{T_A})\, |. \tag{5.7}$$

In particular, the number of genera with discriminant $-\det(T_A)$ is given by $\frac{1}{2}|O(D_{T_A})|$.

Proof. The first equality (5.7) follows from the comparison of the Shioda-Mitani formula (5.6) and Propositions 5.1, 5.3, 5.6. The group of proper equivalence classes of primitive positive-definite rank 2 even oriented lattices with determinant $\det(T_A)$ is canonically isomorphic to the group $\mathcal{C}(\mathcal{O})$ (see [4] Theorem 7.7). Hence the second assertion follows from the fact that all proper genera in a given class group consist of the same number of classes. □

Of course, Corollary 5.11 can be proved directly without going through decompositions of Abelian surfaces. Shioda-Mitani's formula is extended as follows.

Corollary 5.12. *Let A be an Abelian surface as in Theorem 5.10 and suppose that $\det(T_A) \neq 3, 4$. For a natural number $N > 1$ let A_N be the Abelian surface as in Corollary 5.9 and $\mathcal{O}_N$ be the order with discriminant $d(\mathcal{O}_N) = -\det(T_A(N))$. Then*

$$\widetilde{\delta}(A_N) = 2^{\tau(N)} \cdot |\, \mathcal{C}(\mathcal{O}_N)\, |.$$

Proof. By the assumption, we have $\widetilde{\delta}(A) = |\mathcal{C}(\mathcal{O})|$ for the order $\mathcal{O}$ with $d(\mathcal{O}) = -\det(T_A)$. Then our assertion follows from the comparison of the equations (5.4), (5.5) and [4] Corollary 7.28. □

An ideal-theoretic proof of this corollary is also available.

5.3 Abelian surfaces with decomposition number 1

We shall study Abelian surfaces with $\rho(A) = 4$ and $\delta(A) = 1$. Such Abelian surfaces can be classified as follows :

(i) T_A is primitive, $\widetilde{\delta}(A) = 1$.

(ii) T_A is primitive, $\widetilde{\delta}(A) = 2$.

(iii) T_A is not primitive, $\widetilde{\delta}(A) = 2$.

In the below we see that there are exactly thirteen, twenty-nine, four Abelian surfaces in the classes (i), (ii), (iii) respectively.

Example 5.13. Let A be an Abelian surface such that $\rho(A) = 4$ and $\mathrm{Dec}(A) = \{(E, E)\}$ for an elliptic curve E. Then E is isomorphic to one of the following thirteen elliptic curves :

$$\begin{aligned} & \left\{ E(\frac{1+\sqrt{\Delta}}{2}) \;\Big|\; \Delta = -3, -7, -11, -19, -27, -43, -67, -163 \right\} \\ \sqcup & \left\{ E(\sqrt{\Delta}) \;\Big|\; \Delta = -1, -2, -3, -4, -7 \right\}, \end{aligned}$$

where $E(\tau) = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$.

Proof. By Corollary 5.8 and Shioda-Mitani formula (5.6), we have $\widetilde{\delta}(A) = 1$ if and only if T_A is primitive and the class number $|\mathcal{C}(\mathcal{O})|$ of the corresponding order $\mathcal{O}$ is equal to 1. As a result of Heegner-Baker-Stark's theorem, T_A is one of the following lattices (see [4] Theorem 7.30 (ii)) :

$$\begin{aligned} & \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}, \\ & \begin{pmatrix} 2 & 1 \\ 1 & 10 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 14 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 22 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 34 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 82 \end{pmatrix}. \end{aligned} \tag{5.8}$$

By Shioda-Mitani theory we can determine the elliptic curve E explicitly from the transcendental lattice T_A : see [13], Section 3. □

On the other hand, for those A with primitive T_A and $\delta(A) = 1$, $\widetilde{\delta}(A) = 2$ we have the following.

Example 5.14. There exist natural one-to-one correspondences between the following two sets :

(a) The set of isomorphism classes of Abelian surfaces A with $\rho(A) = 4$ such that T_A is primitive and $\mathrm{Dec}(A) = \{(E_1, E_2)\}$, $E_1 \not\simeq E_2$.

(b) The set of imaginary quadratic order with class number 2.

Proof. For an Abelian surface A in the set (a), we associate the order $\mathcal{O}$ with discriminant $d(\mathcal{O}) = -\det(T_A)$. By Shioda-Mitani formula we have $|\mathcal{C}(\mathcal{O})| = 2$. Then T_A corresponds to the non-trivial element of the class group $\mathcal{C}(\mathcal{O})$. □

Imaginary quadratic fields with class number 2 are also classified by [10], [14]. As a result, imaginary quadratic orders with class number 2 are given by the following twenty-nine discriminants :

$$\begin{aligned} -d(\mathcal{O}) \quad = \quad & 15, 20, 24, 32, 35, 36, 40, 48, 51, 52, 60, 64, 75, 88, 91, 99, \\ & 100, 112, 115, 123, 147, 148, 187, 232, 235, 267, 403, 427, 748. \end{aligned}$$

Enumeration of the non-principal forms is a rather straightforward task and is left to the reader.

Example 5.15. Let A be an Abelian surface with $\rho(A) = 4$ and assume that T_A is *not* primitive. If $\delta(A) = 1$, then A is isomorphic to one of the following four Abelian surfaces :

$$E(\sqrt{-1}) \times E(2\sqrt{-1}), \qquad E(\tau_1) \times E(2\tau_1),$$
$$E(\tau_1) \times E(3\tau_1), \qquad E(\tau_2) \times E(2\tau_2),$$

where $\tau_1 = \frac{1+\sqrt{-3}}{2}$, $\tau_2 = \frac{1+\sqrt{-7}}{2}$, and $E(\tau) = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$.

Proof. As T_A is not primitive, we see from Corollary 5.8 that $\widetilde{\delta}(A) = 2$. First consider the case $T_A \simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}$ with $n > 1$. Since the actions of $SO(T_A)$ preserve the fibres of the determinant homomorphism (5.1), we have $\tau(n) = 1$ so that $SO(T_A)$ is a normal subgroup of $O(D_{T_A})$ by Lemma 5.2. Thus we have $|O(D_{T_A})| = 8$, which implies by Theorem 6.7 that $n = 2$. In the case of $T_A \simeq \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}$ with $n > 1$, we have $n = 2, 3$ by similar argument.

Next consider the case $T_A \not\simeq \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}, \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}$. Let B be the Abelian surface such that T_B is primitive and $T_B(n)$ is properly equivalent to T_A, $n > 1$. By Proposition 5.9 the number $\widetilde{\delta}(B)$ divides $\widetilde{\delta}(A) = 2$ so that $\widetilde{\delta}(B) = 1$ or 2. However, the case that $\widetilde{\delta}(B) = 2$ is impossible by Corollary 5.9 (2), (3). Thus we have $\widetilde{\delta}(B) = 1$ so that T_B is one of the lattices in the list (5.8). It follows from Corollary 5.12 that $n = 2$ and $T_B = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ is the only case. ∎

6 Order of isometry group of discriminant form

The aim of this section is to calculate the order of the group $O(D_L)$ for a rank 2 even lattice L. This section may be read independently of the previous sections. By the orthogonal decomposition $D_L = \oplus_p D_p$ where D_p is the p-component for the prime number p, we have the canonical decomposition

$$O(D_L) = \bigoplus_p O(D_p). \tag{6.1}$$

Thus we first calculate $|O(D_p)|$ for each prime number p, and then put them together to $|O(D_L)|$. Throughout this section, a *finite quadratic form* means a finite Abelian group D endowed with a quadratic form $q : D \to \mathbb{Q}/2\mathbb{Z}$ such that the associated bilinear form $b : D \times D \to \mathbb{Q}/\mathbb{Z}$ is non-degenerate. For a finite quadratic form (D, q) on a p-group D with $p \neq 2$, we identify q with the bilinear form b.

6.1 Local calculations

By the classification of finite quadratic forms [16], a finite quadratic form on an Abelian p-group of length 2 is isometric to one of the following forms :

$$\begin{aligned}
A_{p,k}^{\theta,\theta'} &= \begin{pmatrix} \theta p^{-k} & 0 \\ 0 & \theta' p^{-k} \end{pmatrix} \quad \text{on } (\mathbb{Z}/p^k\mathbb{Z})^2, \quad \theta,\theta' \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2, \\
B_{p,l,k}^{\theta,\theta'} &= \begin{pmatrix} \theta p^{-l} & 0 \\ 0 & \theta' p^{-k} \end{pmatrix} \quad \text{on } \mathbb{Z}/p^l\mathbb{Z} \oplus \mathbb{Z}/p^k\mathbb{Z}, \quad l > k, \quad \theta,\theta' \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2, \\
V_k &= \begin{pmatrix} 2^{1-k} & 2^{-k} \\ 2^{-k} & 2^{1-k} \end{pmatrix} \quad \text{on } (\mathbb{Z}/2^k\mathbb{Z})^2, \\
U_k &= \begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix} \quad \text{on } (\mathbb{Z}/2^k\mathbb{Z})^2.
\end{aligned}$$

The following proposition is essentially a consequence of successive approximation.

Proposition 6.1. *Let (D,q) be a finite quadratic form on a p-group with no direct summand of order 2. For the finite quadratic form*

$$(\overline{D},\bar{q}) = (D/N,\ p\cdot q), \quad N = \{x \in D,\ px = 0\},$$

the natural homomorphism

$$\kappa : O(D,q) \longrightarrow O(\overline{D},\bar{q})$$

is surjective.

Proof. Note that $(\overline{D},\bar{q})$ is well-defined as a finite quadratic form. A comparison of the classification of $\mathbb{Z}_p$-lattices and that of finite quadratic forms (cf. [11] Proposition 1.8.1) enables us to find an even $\mathbb{Z}_p$-lattice L such that $(D_{L(p)}, q_{L(p)}) \simeq (D,q)$. Then the homomorphism κ is identified with the homomorphism

$$\kappa' : O(D_{L(p)}, q_{L(p)}) = O(\frac{1}{p}L^\vee/L,\ p\cdot q_L) \to O(\frac{1}{p}L^\vee/\frac{1}{p}L,\ p^2\cdot q_L) \simeq O(D_L, q_L).$$

Consider the natural homomorphisms $O(L) \to O(D_L)$ and $O(L(p)) \to O(D_{L(p)})$, which are surjective by Corollary 1.9.6 of [11]. Since the diagram

$$\begin{array}{ccc} O(L(p)) & = & O(L) \\ \downarrow & & \downarrow \\ O(D_{L(p)}) & \xrightarrow{\kappa'} & O(D_L), \end{array}$$

commutes, we see that κ' is surjective. □

Thus natural reduction homomorphisms

$$O(A_{p,k}^{\theta,\theta'}) \to O(A_{p,k-1}^{\theta,\theta'}), \quad O(B_{p,l,k}^{\theta,\theta'}) \to O(B_{p,l-1,k-1}^{\theta,\theta'}), \quad \cdots$$

are defined and are surjective if $p \neq 2$ or $k \geq 2$.

Lemma 6.2. *We have the following isomorphisms.*

(1) $\mathrm{Ker}(O(A^{\theta,\theta'}_{p,k}) \twoheadrightarrow O(A^{\theta,\theta'}_{p,k-1})) \simeq \mathbb{Z}/p\mathbb{Z}$ *, where* $k \geq 2$ *if* $p \neq 2$*, and* $k \geq 3$ *if* $p = 2$*.*

(2) $\mathrm{Ker}(O(B^{\theta,\theta'}_{p,l,k}) \twoheadrightarrow O(B^{\theta,\theta'}_{p,l-1,k-1})) \simeq \mathbb{Z}/p\mathbb{Z}$ *, where* $k \geq 2$ *if* $p \neq 2$*, and* $k \geq 3$ *if* $p = 2$*.*

(3) $\mathrm{Ker}(O(V_k) \twoheadrightarrow O(V_{k-1})) \simeq \mathbb{Z}/2\mathbb{Z}$ *where* $k \geq 2$*.*

(4) $\mathrm{Ker}(O(U_k) \twoheadrightarrow O(U_{k-1})) \simeq \mathbb{Z}/2\mathbb{Z}$ *where* $k \geq 2$*.*

Proof. We prove only the assertion (2). Other assertions can be proved analogously and are left to the reader. Let $p \neq 2$. An isometry $\gamma \in O(B^{\theta,\theta'}_{p,l,k})$ contained in the kernel of the reduction $O(B^{\theta,\theta'}_{p,l,k}) \to O(B^{\theta,\theta'}_{p,l-1,k-1})$ is represented as

$$\gamma = \begin{pmatrix} 1+p^{l-1}a & p^{l-1}b \\ p^{k-1}c & 1+p^{k-1}d \end{pmatrix}, \quad a,b,c,d \in \mathbb{Z}/p\mathbb{Z}.$$

Since γ preserves the quadratic form, we have

$$ {}^t\gamma \begin{pmatrix} \theta & 0 \\ 0 & p^{l-k}\theta' \end{pmatrix} \gamma \equiv \begin{pmatrix} \theta & 0 \\ 0 & p^{l-k}\theta' \end{pmatrix} \mod \begin{pmatrix} p^l\mathbb{Z} & p^l\mathbb{Z} \\ p^l\mathbb{Z} & p^l\mathbb{Z} \end{pmatrix}. \tag{6.2}$$

Then trivial calculation shows that $a = d = 0$ and $\theta b + \theta' c = 0$ so that the kernel is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. For $p = 2$, we need to replace (6.2) by the equation

$$ {}^t\gamma \begin{pmatrix} \theta & 0 \\ 0 & 2^{l-k}\theta' \end{pmatrix} \gamma \equiv \begin{pmatrix} \theta & 0 \\ 0 & 2^{l-k}\theta' \end{pmatrix} \mod \begin{pmatrix} 2^{l+1}\mathbb{Z} & 2^l\mathbb{Z} \\ 2^l\mathbb{Z} & 2^{l+1}\mathbb{Z} \end{pmatrix}.$$

□

Lemma 6.3. *We have the following isomorphisms.*

(1) $\mathrm{Ker}(O(A^{\theta,\theta'}_{2,2}) \twoheadrightarrow O(A^{\theta,\theta'}_{2,1})) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$*.*

(2) $\mathrm{Ker}(O(B^{\theta,\theta'}_{2,l,2}) \twoheadrightarrow O(B^{\theta,\theta'}_{2,l-1,1})) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$*.*

(3) *If* $p \neq 2$*, then* $\mathrm{Ker}(O(B^{\theta,\theta'}_{p,l,1}) \twoheadrightarrow O(B^{\theta,\theta'}_{p,l-1,0}))$ *is the dihedral group of order* $2p$*.*

Proof. We prove only the assertion (3). An isometry $\gamma \in O(B^{\theta,\theta'}_{p,l,1})$ contained in the kernel is represented as

$$\gamma = \begin{pmatrix} 1+p^{l-1}a & p^{l-1}b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{Z}/p\mathbb{Z}.$$

By the isometry condition we have

$$d^2 = 1, \quad b = -\theta'' dc, \quad a = -2^{-1}\theta'' c^2,$$

where $\theta'' = \theta^{-1}\theta'$. So there are ambiguities of $d \in \{\pm \mathrm{id}\}$ and $c \in \mathbb{Z}/p\mathbb{Z}$. It follows that the kernel is generated by

$$\eta = \begin{pmatrix} 1 - 2^{-1}\theta'' p^{l-1} & -\theta'' p^{l-1} \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with the relations $\eta^p = \sigma^2 = (\eta\sigma)^2 = 1$. □

Lemma 6.4. *Let $\chi_p(a) = \left(\frac{a}{p}\right)$ be the Legendre symbol for $p \neq 2$.*

(1) *For $p \neq 2$, $O(A_{p,1}^{\theta,\theta'})$ is the dihedral group of order $2(p - \chi_p(-\theta\theta'))$.*
(2) *We have*

$$O(A_{2,1}^{\theta,\theta'}) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } \theta\theta' \equiv 1 \mod 4, \\ \{1\}, & \text{if } \theta\theta' \equiv -1 \mod 4. \end{cases}$$

(3) *We have*

$$O(B_{2,l,1}^{\theta,\theta'}) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } l = 2, 3, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } l \geq 4. \end{cases}$$

(4) *$O(V_1)$ is the symmetric group $\mathfrak{S}_3$.*
(5) *$O(U_1) \simeq \mathbb{Z}/2\mathbb{Z}$.*

Proof. See Theorem 11.4 of [15] for the assertion (1). The verifications of the assertions (2), (4), (5) are straightforward. We prove (3). Let $\gamma = \begin{pmatrix} a & 2^{l-1}b \\ c & d \end{pmatrix} \in O(B_{2,l,1}^{\theta,\theta'})$, where $a \in \mathbb{Z}/2^l\mathbb{Z}$ and $b, c, d \in \mathbb{Z}/2\mathbb{Z}$. If we denote $\theta'' := \theta^{-1}\theta'$, the isometry condition for γ is the following equations :

$$a^2 + c^2\theta''2^{l-1} \equiv 1 \qquad \mod 2^{l+1}, \tag{6.3}$$
$$b^2 2^{l-1} + d^2\theta'' \equiv \theta'' \qquad \mod 4, \tag{6.4}$$
$$ab + cd\theta'' \equiv 0 \qquad \mod 2. \tag{6.5}$$

When $l = 2$, we see that $a = \pm 1$, $b = c = 0$, and $d = 1$. When $l \geq 3$, we have $d = 1$ by (6.4) and $b = c$ by (6.5). There are two possibilities for c : 0 or 1. If $c = 0$, then $a = \pm 1$ satisfy the equation (6.3). If $c = 1$, then the equation (6.3) is written as

$$a^2 = 1 - 2^{l-1}\theta'' \mod 2^{l+1}. \tag{6.6}$$

When $l = 3$, (6.6) does not have solution. When $l = 4$, (6.6) has solutions $a = \pm(1 + 4\theta'')$. When $l \geq 5$, (6.6) has solutions $a = \pm(1 - 2^{l-2}\theta'')$. □

From Lemmas 6.2, 6.3, and 6.4 we obtain the following results.

Proposition 6.5. *Let $p \neq 2$ and $k \geq 1$.*

(1) *We have $|O(A_{p,k}^{\theta,\theta'})| = 2 \cdot p^{k-1} \cdot (p - \chi_p(-\theta\theta'))$.*
(2) *We have $|O(B_{p,l,k}^{\theta,\theta'})| = 4 \cdot p^k$.*

Proposition 6.6. *We have the following equalities.*

(1)

$$|O(A_{2,k}^{\theta,\theta'})| = \begin{cases} 2^k, & -\theta\theta' \equiv 1 \mod 4,\ k \geq 2 \\ 2^{k+1}, & -\theta\theta' \equiv -1 \mod 4,\ k \geq 2 \\ 1, & -\theta\theta' \equiv 1 \mod 4,\ k = 1 \\ 2, & -\theta\theta' \equiv -1 \mod 4,\ k = 1 \end{cases}$$

(2)

$$|O(B_{2,l,k}^{\theta,\theta'})| = \begin{cases} 2^{k+1}, & l-k \le 2,\ k \ge 2 \\ 2^{k+2}, & l-k \ge 3,\ k \ge 2 \\ 2, & l-k \le 2,\ k=1 \\ 4, & l-k \ge 3,\ k=1 \end{cases}$$

(3) $|O(V_k)| = 2^k \cdot 3$.
(4) $|O(U_k)| = 2^k$.

6.2 Global results

Let L be an even lattice of rank 2. Denote by (n,m) the invariant factor of $L \subset L^\vee$. That is, we have $n|m$ and there is a basis $\{v_1, v_2\}$ of $L^\vee$ such that $L = \langle nv_1, mv_2\rangle$. Let

$$\begin{aligned} n &= p_1^{e_1} \cdots p_\alpha^{e_\alpha} \cdot q_1^{f_1} \cdots q_\beta^{f_\beta}, \\ m &= p_1^{e_1} \cdots p_\alpha^{e_\alpha} \cdot q_1^{f_1'} \cdots q_\beta^{f_\beta'} \cdot r_1^{g_1} \cdots r_\gamma^{g_\gamma}, \quad f_i' > f_i, \end{aligned}$$

be the prime decompositions of n and m. As groups, the p-components D_p of the discriminant group D_L are as follows :

$$D_{p_i} \simeq (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^2, \quad D_{q_i} \simeq \mathbb{Z}/q_i^{f_i}\mathbb{Z} \oplus \mathbb{Z}/q_i^{f_i'}\mathbb{Z}, \quad D_{r_i} \simeq \mathbb{Z}/r_i^{g_i}\mathbb{Z}.$$

For a prime number p dividing n, put

$$\varepsilon_p := \begin{cases} -p^{-2e_i} \cdot \det(L), & p = p_i. \\ 0, & p = q_i. \end{cases}$$

When $D_p \simeq A_{p,e}^{\theta,\theta'}$ as a quadratic form, we have $-\theta\theta' \equiv \varepsilon_p \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$. When $D_2 \simeq U_e$ (resp. V_e), we have $\varepsilon_2 \equiv 1$ (resp. 5) $\mod 8$.

For an odd prime number p and $\varepsilon \in \mathbb{F}_p^\times$, let $\chi_p(\varepsilon) = \left(\frac{\varepsilon}{p}\right)$ be the Legendre symbol. We put $\chi_p(0) := 0$. For a natural number $n \equiv 1 \mod 4$, we define

$$\chi_2(n) = \begin{cases} 1, & n \equiv 1 \mod 8, \\ -1, & n \equiv 5 \mod 8. \end{cases}$$

For a natural number N let $\widetilde{\tau}(N) := \tau(N)$ if $N > 1$ and $\widetilde{\tau}(1) := 0$. Then we have

$$\widetilde{\tau}(n) + \widetilde{\tau}(n^{-1}m) = \alpha + 2\beta + \gamma.$$

We are now in a position to express the formula for $|O(D_L)|$.

Theorem 6.7. *Let $L, (n,m), \chi_p$, and $\widetilde{\tau}$ be as above.*

(1) *If D_2 is either trivial or U_k or V_k, or equivalently if $L \simeq M(2^e)$ for an even lattice M with* $\det(M)$ *odd, then*

$$|O(D_L)| = 2^{\widetilde{\tau}(n)+\widetilde{\tau}(n^{-1}m)} \cdot n \cdot \prod_{p|n}\Big(1 - \frac{\chi_p(\varepsilon_p)}{p}\Big).$$

(2) *If* $D_2 \simeq A_{2,k}^{\theta,\theta'}$*, or equivalently if* $L \simeq M(2^e)$ *for an odd lattice* M *with* $\det(M)$ *odd, then*

$$|O(D_L)| = C \cdot 2^{\widetilde{\tau}(2^{-1}n)+\widetilde{\tau}(n^{-1}m)} \cdot n \cdot \prod_{\substack{p|n \\ p\neq 2}} \Big(1 - \frac{\chi_p(\varepsilon_p)}{p}\Big),$$

where $C = 1$ *if* $\varepsilon_2 \equiv -1 \mod 4$, $C = \frac{1}{2}$ *if* $\varepsilon_2 \equiv 1 \mod 4$.

(3) *If* $D_2 \simeq B_{2,l,k}^{\theta,\theta'}$*, or equivalently if* $L \simeq M(2^e)$ *for an odd lattice* M *with* $\det(M)$ *even, then*

$$|O(D_L)| = C \cdot 2^{\widetilde{\tau}(2^{-1}n)+\widetilde{\tau}(n^{-1}m)} \cdot n \cdot \prod_{\substack{p|n \\ p\neq 2}} \Big(1 - \frac{\chi_p(\varepsilon_p)}{p}\Big),$$

where $C = 1$ *if* $l - k \geq 3$, $C = \frac{1}{2}$ *if* $l - k \leq 2$.

Proof. This follows immediately from the results of the previous section 6.1. Note that D_2 is never cyclic. $\square$

From section 6.1 we also deduce the following.

Proposition 6.8. *Let* L *be a rank* 2 *even lattice and* $n > 1$ *be a natural number.*

(1) *The number* $|O(D_{L(n)})|$ *is divisible by the number* $|O(D_L)|$.

(2) *If* n *is coprime to* $\det(L)$*, we have*

$$|O(D_{L(n)})| = |O(D_L)| \cdot 2^{\tau(n)} \cdot n \cdot \prod_{p|n} \left(1 - \frac{\chi_p(-\det(L))}{p}\right).$$

(3) *If* L *is primitive and* $n|\det(L)^a$ *for some* $a \in \mathbb{Z}_{>0}$*, we have*

$$|O(D_{L(n)})| = |O(D_L)| \cdot 2^{\tau(n)} \cdot n.$$

Proof. The assertion (1) follows from Proposition 6.1.

(2) We identify the $\mathbb{Z}$-modules underlying L and $L(n)$ in a natural way. Since n is coprime to $|D_L|$, we have the orthogonal decomposition

$$D_{L(n)} = (L^\vee/L) \oplus (n^{-1}L/L).$$

Hence we have

$$O(D_{L(n)}) = O(D_L) \oplus \bigoplus_{p|n} O(p^{-e}L/L),$$

where $n = \prod p^e$ is the prime decomposition of n. If p is odd, then $p^{-e}L/L \simeq A_{p,e}^{1,\det(L)}$. If $p = 2$, we have $2^{-e}L/L \simeq U_e$ or V_e according to $-\det(L) \equiv 1$ or 5 mod 8.

(3) Let $D_L = \oplus_p D_p$ be the decomposition into p-components. For $p \neq 2$, D_p is cyclic. On the other hand, D_2 is either trivial or $A_{2,1}^{\theta,\theta'}$ or $B_{2,l,1}^{\theta,\theta'}$. Thus our claim follows from Lemmas 6.2 and 6.3. $\square$

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